The spectrum of equivariant Kasparov theory for cyclic groups of prime order

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Reference:
Ivo Dell'Ambrogio and Ralf Meyer, arXiv:2009.05424 (Sept 2020)

1. The equivariant Kasparov category

G : a 2nd countable locally compact group
$\rightsquigarrow$ KK^G : the **G-equivariant Kasparov category** (Kasparov 1988)

- Objects: separable complex G-C*-algebras
- Hom sets: $Hom(A, B) = KK_0^G(A, B)$
  Kasparov cycles $/\!\sim$, or generalized equiv. $*$-homomorphisms, or ...
- composition & a symmetric monoidal structure: 'Kasparov product'

(Meyer-Nest 2006)

KK^G is a **tensor triangulated category**, that is:

- additive category: can sum morphisms and objects (usual direct sums)
- suspension functor $\Sigma \colon A \mapsto C_0(\mathbb{R}) \otimes A$ (invertible by Bott: $\Sigma^2 \cong \mathrm{Id}$)
- triangles: $\Sigma C \to A \to B \to C$ (mapping cones / cpc-split extensions)
- bi-exact tensor product: $A \otimes_{\min} B$ with diagonal G-action

2. Tensor triangulated categories

This algebraic structure captures for example: $\forall D \in KK^G$

- **Homological algebra:** get LES from triangles $\Sigma C \xrightarrow{\partial} A \to B \to C$:

$$\ldots \to Hom(D, \Sigma C) \xrightarrow{\partial_*} Hom(D, A) \to Hom(D, B) \to Hom(D, C) \xrightarrow{\partial_*} \ldots$$

$$\ldots \leftarrow Hom(\Sigma C, D) \xleftarrow{\partial^*} Hom(A, D) \leftarrow Hom(B, D) \leftarrow Hom(C, D) \xleftarrow{\partial^*} \ldots$$

- **Bootstrap-like constructions:** $\mathcal{S}$ any set of objects:
  Thick($\mathcal{S}$) := closure of $\mathcal{S}$ under $\Sigma^{\pm}$, sums, mapping cones, retracts, and isomorphic objects.
  Loc($\mathcal{S}$) := as above + closed under *infinite* direct sums. coproducts

  Both constructions yield (full) triangulated subcategories.

  Thick$_{\otimes}(\mathcal{S})$, Loc$_{\otimes}(\mathcal{S})$: variants closed under tensoring with any objects
  $\rightsquigarrow$ these are (thick, localizing) *tensor ideals*. $A \in$ & B any $\Rightarrow A \otimes B \in$

3. The Balmer spectrum

Can also 'do geometry':

supp ΣA = supp A

(Balmer 2005)

Every (essentially small) tensor triangulated category $\mathcal{T}$ admits a 'universal support theory', namely:

supp$(A \oplus B)$ = supp $A \cup$ supp B

- A topological space $\mathrm{Spc}(\mathcal{T})$, the **spectrum** of $\mathcal{T}$. $\otimes$ $\cap$
- For each $A \in \mathcal{T}$, a closed subset $\mathrm{supp}(A) \subset \mathrm{Spc}(\mathcal{T})$, its **support**.
- This data yields a rough geometric classification of objects: $\mathrm{Thick}_{\otimes}(A) = \mathrm{Thick}_{\otimes}(B) \quad \Leftrightarrow \quad \mathrm{supp}(A) = \mathrm{supp}(B)$

Examples:

- (Thomason 1997) V an quasi-compact and quasi-separated scheme, $\mathcal{T} = D^{perf}(V) \quad \rightsquigarrow \quad \mathrm{Spc}(\mathcal{T}) \cong V$.
  In particular for $V = \mathrm{Spec}(R) \quad \rightsquigarrow \quad \mathrm{Spc}(D^b(\text{proj-}R)) \cong \mathrm{Spec}(R)$.
- (Benson-Carlson-Rickard 1997) G a finite group, $\mathrm{char}(k) \mid |G|$, $\mathcal{T} = \mathrm{stmod}(kG) \quad \rightsquigarrow \quad \mathrm{Spc}(\mathcal{T}) \cong \mathrm{Proj}(H^*(G; k))$.

4. So, what about $\mathcal{T} = KK^G$?

A very nice characterisation of the Baum-Connes assembly map:

(Meyer-Nest 2006)

The inclusion functor of the following subcategory Cell^H Cell^G

$$\mathcal{CI} := \mathrm{Loc}_{(\otimes)}\Big(\bigcup_{H \leq G \text{ compact}} \mathrm{Ind}_H^G(\cancel{KK}^H) \Big) \quad \subset \quad \cancel{KK}^G$$

Cell^H

has a right adjoint $A \mapsto \tilde{A} \in \mathcal{CI}$. Applying $K_*(G \ltimes -)$ to the counit of adjunction $\varepsilon_A \colon \tilde{A} \to A$ we get the Baum-Connes assembly map with coefficients in $A \in KK^G$.

$\mathrm{Spc}(KK^H)$

Tantalizingly: $\mathrm{Spc}(-)$ functorial

$\mathrm{Spc}(\mathrm{Cell}^H_c)$

(D. 2008)

If the natural map $(\mathrm{Res}_H^G)_H^* \colon \bigcup_{H \text{ cpt}} \mathrm{Spc}(KK^H) \to \mathrm{Spc}(KK^G)$ is surjective, we have $\mathcal{CI} = KK^G$, hence $\tilde{A} \overset{\simeq}{\to} A$, hence BC holds for G and all A.

5. Towards to spectrum of Kasparov theory

Unfortunately, the computation of $\mathrm{Spc}(KK^G)$ seems well out of reach!
Only general result known:

(Balmer 2010)

For any (essentially small) tensor triangulated $\mathcal{T}$, there is a natural continuous map

$$\rho_{\mathcal{T}} \colon \mathrm{Spc}(\mathcal{T}) \longrightarrow \mathrm{Spec}(\mathrm{End}_{\mathcal{T}}(\mathbf{1}))$$

to the Zariski spectrum of the endomorphism ring of the tensor unit object $\mathbf{1}$. It is surjective as soon as $\mathrm{End}_{\mathcal{T}}(\mathbf{1})_*$ is noetherian.

Corollary

For G a compact Lie group, we have a surjective map

$$\mathrm{Spc}(KK^G) \twoheadrightarrow \mathrm{Spec}(\mathrm{R}_{\mathbb{C}}(G))$$

onto the Zariski spectrum of its complex character ring.

$\mathbb{1} = \mathbb{C} \in KK^G = \mathcal{T}$

$\mathrm{End}(\mathbb{1}) \cong R_{\mathbb{C}}(G)$

$\mathrm{End}(\mathbb{1})_* \cong \underline{R_{\mathbb{C}}(G)[\beta^{\pm 1}]}$

noeth

6. Bootstrap categories are nicer

Main technical difficulties:

- KK^G has no good **generation properties**.
- KK^G has (countable) infinite direct sums, but $\mathrm{Spc}(-)$ is best for (sub-)categories of **compact** and **dualizable** objects A: those which
  - satisfy $Hom(A, \bigoplus_i B_i) \cong \bigoplus_i Hom(A, B_i)$ ← compact
  - and have a tensor-dual $A^\vee$: $Hom(A \otimes B, C) \cong Hom(B, A^\vee \otimes C)$. ← dual'ble

Definition: G-cell algebras G-equiv bootstrap constr.

$\color{red}{Cell^G} := \mathrm{Loc}(\{C(G/H) \colon H \leq G \text{ a closed subgroup}\}) \quad \subset KK^G$

- $Cell^1$ is the usual Rosenberg-Schochet bootstrap category.
- For G compact, $Cell^G$ is again a tensor triangulated category, and: $Cell^G$
  - it is 'countably compactly-rigidly generated'. $C(G/H)$ dual'ble + cpt $\cup$
  - its compact and dualizable objects agree $\rightsquigarrow$ they form a nice $\boxed{\text{ttc } \color{red}{Cell^G_c}}$.

else wrong

7. The spectrum of compact G-cell algebras

$\exists$ cont. section $\qquad$ $G=1 \Rightarrow$ bij

(D. 2010)

For G finite, the map $\rho\colon \mathrm{Spc}(\mathit{Cell}_c^G) \longrightarrow \mathrm{Spec}(\mathrm{R}_{\mathbb{C}}(G))$ is split surjective.

(D.-Meyer 2020) $\qquad$ $G \cong \mathbb{Z}/p\mathbb{Z}$

For G cyclic of prime order, the map $\rho\colon \mathrm{Spc}(\mathit{Cell}_c^G) \xrightarrow{\sim} \mathrm{Spec}(\mathrm{R}_{\mathbb{C}}(G))$ is injective, hence a homeomorphism.

From now on, ideas for the proof. Set $G \cong \mathbb{Z}/p\mathbb{Z}$ for a prime p.
Recall:

$$\mathrm{R}_{\mathbb{C}}(G) \cong \mathbb{Z}[\hat{G}] \cong \mathbb{Z}[x]/(x^p - 1)$$

and $x^p - 1$ has two irreducible factors:

$$x - 1 \quad \text{and} \quad \Phi_p = 1 + x + \ldots + x^{p-1}.$$

Modding them out in turn:

$\ni$ a prim. p-th root of 1 $\mathbb{C} \cup$

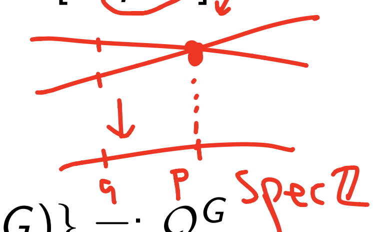
$$\mathbb{Z} \xleftarrow{\text{mod } x-1} \underbrace{\mathbb{Z}[x]/(x^p - 1)}_{\mathrm{R}_{\mathbb{C}}(G)} \xrightarrow{\text{mod } \Phi_p} \mathbb{Z}[x]/(\Phi_p) := \mathbb{Z}[\vartheta]$$

Two irreducible components, their intersection is the unique closed point over p. By inverting p on the RHS, get a disjoint union decomposition:

$\exists!$

$$\operatorname{Spec}\mathbb{Z} \rightarrowtail \operatorname{Spec}\mathrm{R}_{\mathbb{C}}(G) \leftarrowtail \operatorname{Spec}\mathbb{Z}[\vartheta, p^{-1}]$$

Now, lift 'the same' decomposition to $Cell^G$, as follows:



$\mathbb{C} \mapsto \cdots \mathrm{Ind} \dashrightarrow C(G)$

$$Cell^1 \xleftarrow{\mathrm{Res}^G_1} Cell^G \xrightarrow{\text{localisation}} Cell^G/\mathrm{Loc}\{C(G)\} =: \mathcal{Q}^G$$

$=$ usual bootstrap $\subset KK$ $\subset KK^G$ $\mathbb{C}, C(G)$ generate $Cell^G$ gen. by $\mathbb{C}$

9. Computation for $G \cong \mathbb{Z}/p\mathbb{Z}$

Restrict these two tensor-exact functors to compact objects and apply $\mathrm{Spc}(-)$ to get the top row:

$\mathbb{Z}/p \times \mathbb{Z}/p$ $\quad$ $\mathbb{Z}/p \times \mathbb{Z}/p'$

$$\begin{tikzcd}
\mathrm{Spc}\, Cell_c^1 \arrow[r, tail] \arrow[d, "\rho"', "\cong"] & \mathrm{Spc}\, Cell_c^G \arrow[d, "\rho"'] & \mathrm{Spc}\, \mathcal{Q}_c^G \arrow[l, tail] \arrow[d, "\cong"', "\rho"] \\
\mathrm{Spec}\, \mathbb{Z} \arrow[r, tail] & \mathrm{Spec}\, \mathrm{R}_{\mathbb{C}}(G) & \mathrm{Spec}\, \mathbb{Z}[\vartheta, p^{-1}] \arrow[l, tail]
\end{tikzcd}$$

- ✓ The top row is also a disjoint union decomposition (Balmer 2005+15).
- ✓ The left ρ is known to be bijective (D. 2010). uses the UCT !!!
- ! ✓ $\mathrm{End}(\mathbf{1})_* \cong \mathbb{Z}[\vartheta, p^{-1}, \beta^{\pm 1}]$ in $\mathcal{Q}^G$, computed thanks to Köhler's UCT.
- ✓ In particular, the right square commutes! (2010) for these G !
- ✓ The right ρ is bijective by an abstract criterion (D.-Stanley 2016), since $\mathcal{Q}_c^G = \mathrm{Thick}\{\mathbf{1}\}$ by construction and $\mathrm{End}(\mathbf{1})_*$ is regular as seen. + noeth.

Hence the middle ρ is bijective as well. QED

$\underline{Q = \mathrm{loc}(1)}$, $\mathrm{End}(1)_*$ noeth & reg,

$G = SL_n(\mathbb{C})$ $\qquad$ kk^G $\qquad$ G abstr. group

$\cup$

"Cell^G"